Faculty Working Papers

CONVEX OPTIMIZATION AND LAGRANGE MULTIPLIERS

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Abstract

We show how the duality theorem of linear programming can be used to prove several results on general convex optimization.

Let $f, g_1, \ldots, g_k$ be convex functions defined on a convex subset S of a vector space. Let $T = \{x \in S | g_i(x) \leq 0 \quad 1 \leq i \leq k\}$. We assume throughout that T is non-empty.

We use linear programming theory to explore the relationship between the problem of minimizing $f(x)$ $x \in T$ and the Lagrange dual problem of minimizing $f(x) + \Sigma\lambda_i g_i(x)$ $x \in S$ for suitable $\lambda_i \geq 0$. This is motivated by the work of Duffin [1, 2, 3]. The main tool we shall need is a version of the duality theorem of linear programming [5, theorems 1.1.9 and 1.7.13].

Lemma: Let $A = \{x | Bx \geq b \text{ and } Cx = c\}$. If A is empty, there are $U \geq 0$ and V such that $U^t B + V^t C = \vec{0}$ and $Ub + Vc > 0$. If A is non-empty and every $x \in A$ satisfies $dx \geq e$, then there are $U \geq 0$ and V such that $U^t B + V^t C = d$ and $Ub + Vc \geq e$.

We will not require separating hyperplane theorems or results from semi-infinite programming.

<u>Theorem 1</u>: $f(x) \geq L$ for every $x \in T$ if and only if for every finite $F \subset S$ there are $\lambda_1, \ldots, \lambda_k$ such that $f(x) + \Sigma\lambda_i g_i(x) \geq L$ for every $x \in F$.

<u>Proof</u>: The "if" part is immediate. For the "only if" part it suffices to prove the result for those finite F which contain members of T. Let $F = \{y_1, \ldots, y_N\}$ $y_1 \in T$. For such F the system of equations and inequalities in unknowns $\theta_1, \ldots, \theta_N$

(D)
$$\sum_1^N \theta_i = 1$$

$$\sum_{i=1}^N \theta_i g_j(y_i) \leq 0 \qquad 1 \leq j \leq k$$

$$\theta_i \geq 0$$

has the solution $\theta_1 = 1$. By convexity, if θ_i is a solution to (D), $\Sigma\theta_i y_i \in T$.

Since we are assuming $f(x)\geq L$ for $x\in T$, every solution to (D) must satisfy $\Sigma\theta_i f(y_i)\geq L$. By the lemma there are $\lambda_i\geq 0$ and γ such that

$$\gamma + \sum_{j=1}^{k} \lambda_j(-g_j(y_i)) \leq f(y_i) \qquad 1\leq i\leq N$$

$$\gamma \geq L$$

so
$$f(y_i) + \sum_{j=1}^{k} \lambda_j g_j(y_i)\geq L \text{ for every } y_i\in F. \quad \text{Q.E.D.}$$

Theorem 1 may be used to prove many of the standard results on convex optimization. As an example we prove the Kuhn-Tucker theorem.

Corollary: Suppose $f(x)\geq L$ for $x\in T$ and that there is a y for which $g_i(y)<0$ $1\leq i\leq k$. Then there are $\lambda_1,\ldots,\lambda_k$ such that $f(x)+\Sigma\lambda_i g_i(x)\geq L$ for $x\in S$.

Proof: Let $\delta = \max\{g_i(y)\}$. For $x\in S$ let $A_x=\{(\lambda_1,\ldots,\lambda_k)\,|\,\lambda_i\geq 0$, $f(x)+\Sigma\lambda_i g_i(x)\geq L$, and $-\delta(\Sigma\lambda_i)\leq f(y)-L\}$. For each $x\in S$, A_x is compact. If $H\subset S$ is finite we may use theorem 1 with $F=H\cup\{y\}$ to show $\bigcap_{x\in H} A_x$ is non-empty. Therefore, $\bigcap_{x\in S} A_x$ is non-empty, so suitable Lagrange multipliers exist. Q.E.D.

Arguments of this kind can also be used to give information about when "duality gaps" occur.

Theorem 2. There are no λ_i such that $f(x)+\Sigma\lambda_i g_i(x)\geq L$ for every $x\in S$ if and only if, for every $N\geq 0$, there is an $x\in S$ such that $f(x)<L - N(\max_{1\leq i\leq k} g_i(x))$.

Proof: Take any $\lambda_i\geq 0$ and suppose x exists with the desired property for $N=\Sigma\lambda_i$. Then $f(x)+\Sigma\lambda_i g_i(x)\leq f(x) + (\max g_i(x))\,(\Sigma\lambda_i)<L$. So no suitable λ_i exist.

Conversely, suppose there are no suitable λ_i. For any $N\geq 0$ and any $F=\{y_1,\ldots,y_M\}\subset S$ consider the linear system in unknowns $\lambda_1,\ldots,\lambda_k$:

$$f(y_i) + \sum_1^k \lambda_j g_j(y_i) \geq L \quad y_i \in F$$

$$(F;N) \qquad \Sigma\lambda_i \leq N$$

$$\lambda_i \geq 0 .$$

If, for some N, (F;N) had a solution for every finite FCS, a compactness argument similar to that in the corollary to Theorem 1 would yield suitable multipliers λ_i. Since we are assuming such λ_i do not exist, it must be that for every $N>0$ there is an F such that (F;N) has no solution. By the lemma, if (F;N) has no solution, there are $\theta_1,\ldots,\theta_M \geq 0$ and $\gamma \geq 0$ such that

$$\sum_{i=1}^{M} \theta_i g_j(y_i) - \gamma \leq 0, \quad 1 \leq j \leq k$$

and $\Sigma\theta_i(L-f(y_i)) + \gamma(-N) > 0$. By scaling, we may assume $\sum_1^M \theta_i = 1$, so that

$$\sum_{i=1}^{M} \theta_i g_j(y_i) \leq \gamma < \frac{1}{N}(L - \Sigma\theta_i f(y_i)).$$

If we take $x=\Sigma\theta_i y_i$ $\gamma \geq g_j(x)$, $1 \leq j \leq k$ and $f(x) < L - N\gamma$ follow by convexity of f and g_j. Q.E.D.

Corollary: (Compare [1], cor. 5; [2], thm. 3): Let $h(\epsilon) = \inf\{f(x) \mid g_j(x) \leq \epsilon,$ $1 \leq j \leq k$. If there are $\delta > 0$, L such that $h(x) > L$ for $0 \leq x \leq \delta$, then there are $\lambda_1,\ldots,\lambda_k$ such that $f(x) + \Sigma\lambda_i g_i(x) \geq L$ $x \in S$.

Proof: If there is an x for which $g_j(x) < 0$ the existence of suitable λ_i follows from the Kuhn-Tucker theorem,* so we assume this is not the case. h is a convex monotone function which, on our assumptions, is defined only for non-negative arguments. For $\epsilon \geq \delta$, $h(\epsilon) - L > h(\epsilon) - h(0) \geq \epsilon(\frac{1}{\delta})(h(\delta) - h(0))$. Hence for $x \in S$, $f(x) - L \geq (\max g_i(x))(\min 0, \frac{1}{\delta}(h(\delta) - h(0)))$. (Note that our assumptions

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*We use the Kuhn-Tucker theorem for brevity. The result could be proved from Theorem 2 alone.

imply $\max g_j(x) \geq 0$.) Since the condition given by Theorem 2 fails for $N = \max(0, \frac{1}{\delta}(h(0)-h(\delta))$, suitable λ_i exist.) Q.E.D.

Finally, we use a variation of these techniques to strengthen a recent result of Duffin and Jeroslow [4].

<u>Theorem 3</u>: Let $S=R^n$. Assume that for $\lambda_i>0$, $f(x)+\Sigma\lambda_i g_i(x) \geq L$, $x \in S$. Then there are affine functions $h_i(x)=a_i x+b_i$ $(a_i \in R^n,\ b_i \in R)$ such that $h_i(x) \leq g_i(x)$ and $f(x)+\Sigma\lambda_i h_i(x) \geq L$, $x \in S$.

Proof: For $y \in S$, let $T_y = \{(h_1,\ldots,h_k) \mid h_i$ affine, $h(y) \leq g(y)$, and $f(y)+\Sigma\lambda_i h_i(y) \geq L\}$.

We identify each function $h_i(x)=a_i x+b_i$ with the ordered pair $(a_i,b_i) \in R^{n+1}$. Thus, T_y is a closed subset of $R^{(n+1)k}$. We first show that, for any finite $F \subset S$ $\bigcap_{y \in F} T_y$ is non-empty. A member of $\bigcap_{y \in F} T_y$ would be a solution to the linear inequality system in unknowns $a_1,\ldots,a_k$; $b_1,\ldots,b_k$.

$$a_i y + b_i \leq g_i(y) \qquad 1 \leq i \leq k; \quad y \in F$$

$$\text{(E)} \qquad \sum_{i=1}^{k} \lambda_i (a_i y + b_i) \geq M - f(y) \qquad y \in F$$

By the lemma, (E) has no solution only if there are scalars W_{iy}, $V_y \geq 0$ such that

$$\text{(i)} \qquad \sum_{y \in F} W_{iy} y = \lambda_i \sum_{y \in F} V_y y, \quad 1 \leq i \leq k$$

$$\text{(ii)} \qquad \sum_{y \in F} W_{iy} = \lambda_i \sum_{y \in F} V_y, \quad 1 \leq i \leq k$$

$$\text{(iii)} \qquad \sum_{\substack{1 \leq i \leq k \\ y \in F}} W_{iy} g_i(y) < \sum_{y \in F} V_y (M - f(y)).$$

If there were W,V satisfying (i)-(iii) we could set $V'_y = V_y / \Sigma V_y$ and $W'_{iy} = W_{iy} / \lambda_i \Sigma V_y$ so that (i)-(iii) would be satisfied and, by (ii), $\Sigma V'_y = \sum_{y \in F} W'_{iy} = 1$, $1 \leq i \leq k$. Condition (i) becomes $\Sigma W'_{iy} y = \Sigma V'_y y$. Condition (iii) becomes $\sum_{y \in F} V'_y f(y) + \Sigma\lambda_i (\sum_{y \in F} W'_{iy} g_i(y)) < M$. If $z = \Sigma W'_{iy} y = \Sigma V'_y y$ this implies, by convexity, $f(z)+\Sigma\lambda_i g_i(z) < M$

which contradicts our assumption about the λ_i. Therefore (E) has solutions for every finite F.

To complete the proof we must show $\bigcap_{y\in S} T_y$ is non-empty. Let e_j=jth unit vector. We show that if F contains $\pm e_j$ $1\le j\le n$ and the zero vector, then $\bigcap_{y\in F} T_y$ is bounded. Since each T_y is closed, compactness yields the desired result. For $1\le i\le k$ we must have $h_i(\vec{0})=b_i\le g_i(0)$, $f(\vec{0})+\Sigma\lambda_i b_i\ge L$, $h_i(e_j)=$(jth component of a_i) + $b_i\le g_i(e_j)$, and $h_i(-e_j)\le g_i(-e_j)$. Since all the λ_i are positive this implies bounds on a_i, b_i. Q.E.D.

Our proof of Theorem 3 works for any convex $S\subset R^n$ which includes $\pm e_j$ and $\vec{0}$. By suitable translations, this implies the results for any fully dimensional convex $S\subset R^n$. Further modifications yield the result for arbitrary convex $S\subset R^n$.

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